ON THE DECYCLING OF POWERS AND PRODUCTS OF CYCLES

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ABSTRACT. We calculate exact values of the decycling numbers of $C_m \times C_n$ for $m = 3, 4$, of C_n^2, and of C_n^3.

1. Introduction and definitions

If G is a simple graph, $S \subseteq V(G)$, and $G - S$ is acyclic, then S is a *decycling set* of G. The size of a smallest decycling set is the decycling number $\nabla(G)$ of G. It turns out that, in contrast to the corresponding problem for edges, finding the decycling number can be a quite difficult problem, even for some very simple families of graphs. The problem is NP-complete in general [4]. Much recent work in the area has been focused on calculating the decycling numbers of various families of graphs, e.g. hypercubes, grids, and so on. For a good introduction to and bibliography of recent work of this nature see [1,3]. Most of these results, rather than yielding exact values, are instead in the form of feasible ranges. In this paper we calculate exact decycling numbers for the families $C_3 \times C_n$ for $n \geq 3$, $C_4 \times C_n$ for $n \geq 4$, C_n^2 for $n \geq 4$, and C_n^3 for $n \geq 5$. Note that $G \times H$ is the familiar cartesian product of graphs. Also, the n^{th} power of G, denoted G^n, is defined by $V(G^n) = V(G)$ and

$$E(G^n) = E(G) \cup \{uv | u, v \in V(G) \text{ and } d(u, v) \leq n\}$$

2. Decycling $C_m \times C_n$ for $m \leq n$ and $m = 3, 4$

The calculation of the decycling number of the cartesian product of cycles $C_m \times C_n$ for $m \leq n$ is mentioned in several papers as an important open problem (for instance [1]). We will need the following important theorem from [2] both in this section and throughout the paper:

Theorem 1. *If G is a connected simple graph with maximum degree Δ then*

$$\nabla(G) \geq \frac{|E(G)| - |V(G)| + 1}{\Delta - 1}$$

Corollary 1. $\nabla(C_3 \times C_n) \geq n + 1$.

Theorem 2. $\nabla(C_3 \times C_n) = n + 1$

Proof: We need only show that $\nabla(C_3 \times C_n) \leq n + 1$. Let the three canonical n-cycles of the graph be A_i $1 \leq i \leq 3$ in consecutive cyclic order. Let the n canonical 3-cycles be B_i $1 \leq i \leq n$ in consecutive cyclic order. Let $G = C_3 \times C_n$. Let $C \subseteq V(G)$ be a set of vertices v_i such that v_i is on B_i and v_i and v_{i+1} are not on the same A_j for any i. Then

2000 *Mathematics Subject Classification.* 05C38.
Key words and phrases. decycling graphs, cycles.
I would like to thank Sarah Lynch and her colleagues for providing a congenial work environment. Also I would like to thank my colleague Martha Walker for her interest and encouragement.

$|E(G-C)| = 6n - 4n = 2n$ and $|V(G-C)| = 3n - n = 2n$. Finally, it is clear that $G - C$ is connected, so that it is therefore unicyclic. Thus $C \cup \{v\}$, where v is any vertex on the one remaining cycle of $G - C$, is a decycling set of G with cardinality $n + 1$. □

Theorem 3. $\nabla(C_4 \times C_n) = \left\lceil \frac{3}{2} n \right\rceil$ *for* $n \geq 4$

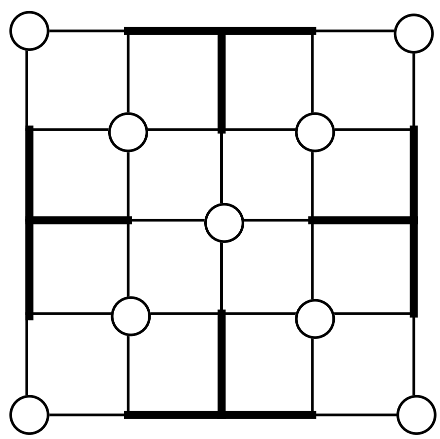
Proof: Let S be a decycling set for $C_4 \times C_n$. Let the n canonical cubes which constitute the graph be q_i for $1 \leq i \leq n$ in consecutive cyclic order. Let $n_i = |V(q_i) \cap S|$. Note that $\nabla(q_i) = 3$ and so

$$3n \leq \sum_{i=1}^{n} n_i = 2|S|$$

and it follows easily that $|S| \geq \left\lceil \frac{3}{2} n \right\rceil$.

Note that in the following diagrams we draw $C_4 \times C_n$ in its standard toroidal embedding with the torus represented as a rectangle with appropriate identifications. We circle vertices which are members of a decycling set, and we draw edges of the subgraph induced by the complement of the decycling set more thickly than the edges which are deleted when the decycling set is deleted. Figure 1 shows a decycling set for $C_4 \times C_4$ of cardinality 6, from which it follows that $\nabla(C_4 \times C_4) = 6$:

Figure 1.



For $C_4 \times C_{2k}$ we augment this graph by inserting $k - 2$ copies of the cylinder shown in Figure 2.

The odd case is similar. For $C_4 \times C_5$ we have the decycling set shown in Figure 3, and then for $C_4 \times C_{2k+1}$ we augment by inserting $k - 2$ copies of the cylinder shown in Figure 2. It is easily seen that these decycling sets have the requisite cardinality. □

3. Decycling C_n^2 for $n \geq 4$

The following is a consequence of Theorem 1:

Corollary 2. $\nabla\left(C_n^2\right) \geq \left\lceil \frac{n+1}{3} \right\rceil$

Figure 2.

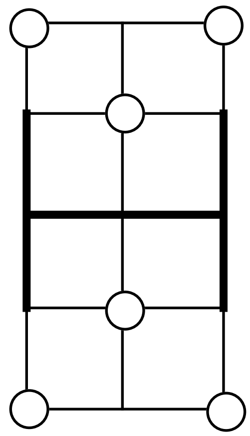
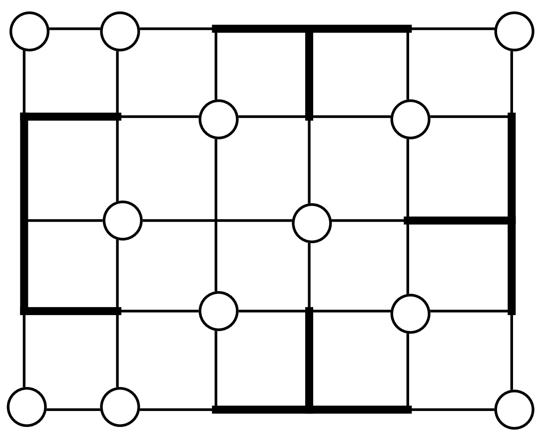


Figure 3.



Theorem 4. $\nabla\left(C_n^2\right) = \begin{cases} \left\lceil \frac{n+1}{3} \right\rceil & n \not\equiv 2 \pmod 3 \\ \left\lceil \frac{n+1}{3} \right\rceil + 1 & n \equiv 2 \pmod 3 \end{cases}$

Proof: Note that for $n \not\equiv 2 \pmod 3$ the lower bound follows from Corollary 2. Let the vertices of C_n^2 be numbered cyclically from 0 to $n-1$. Note that in the remainder of the paper we assume this numbering without special mention.

Case 1: $n \equiv 0 \pmod 3$

Let $S = \{0, 3, \ldots, n-3, n-1\}$. The subgraph induced by $V\left(C_n^2\right) - S$ is the path $\{1, 2, 4, 5, \ldots, n-5, n-4, n-2\}$ and hence S decycles C_n^2. Furthermore $|S| = \frac{n}{3} + 1 = \left\lceil \frac{n+1}{3} \right\rceil$.

Case 2: $n \equiv 1 \pmod 3$

Let $S = \{0, 3, \ldots, n-1\}$. Then the subgraph induced by $V\left(C_n^2\right) - S$ is the path $\{1, 2, 4, 5, \ldots, n-3, n-2\}$ and $|S| = \frac{n-1}{3} + 1 = \left\lceil \frac{n+1}{3} \right\rceil$

Case 3: $n \equiv 2 \pmod 3$

Let $S = \{0, 3, 6, \ldots, n-2, n-1\}$. It is easy to see that S decycles C_n^2 and hence that $\nabla(C_n^2) \leq \left\lceil \frac{n+1}{3} \right\rceil + 1$. Now suppose we have a decycling set S with $|S| = \left\lceil \frac{n+1}{3} \right\rceil = \frac{n+1}{3}$. We first show that S cannot contain two consecutive elements of $V(C_n^2)$. Suppose by way of contradiction that $\{0, 1\} \subset S$. Note that $|S - \{0, 1\}| = \frac{n-5}{3}$ and that $\left| V\left(C_n^2\right) - \{0, 1\} \right| = n - 2$. Let $B_i = \{i, i+1, i+2\}$ and let $n_i = |B_i \cap S|$. Then since each consecutive three elements must contain at least one element of S and each element of S is in three of the B_i's we have

$$n \leq \sum_{i=0}^{n-1} n_i = 3|S| = n + 1$$

However, $n_{-1}, n_0 \geq 2$, so there must be some j with $2 \leq j \leq n-2$ with $n_j = 0$. This is a contradiction to the assumption that S decycles C_n^2.

Now, let $i \in V\left(C_n^2\right)$. Then $\{i+1, i+2\} \cap \left(V\left(C_n^2\right) - S\right) \neq \emptyset$ and $\{i-1, i-2\} \cap \left(V\left(C_n^2\right) - S\right) \neq \emptyset$. Hence the degree of i in the subgraph induced by $V\left(C_n^2\right) - S$ is at least 2. Thus that subgraph contains a cycle, and this contradicts the assumption that S decycles C_n^2. □

4. Decycling C_n^3 for $n \geq 5$

In this case it turns out that the bound on $\nabla(C_n^3)$ given by Theorem 1 is too low in all but a small finite number of cases. The result is:

Theorem 5.

$$\nabla(C_n^3) = \begin{cases} \frac{n+2}{2} & n \equiv 0 \pmod 2 \\ \frac{n+1}{2} & n \equiv 1 \pmod 4 \\ \frac{n+3}{2} & n \equiv 3 \pmod 4 \end{cases}$$

Proof: As above we let $B_i = \{i, i+1, i+2, i+3\}$ and $n_i = |B_i \cap S|$ for a decycling set S.

Case 1: $n \equiv 1 \pmod 4$

First of all, $S = \{0, 1, 2, 5, 6, 9, 10, \ldots, n-4, n-3\}$ decycles C_n^3 and has the advertised cardinality. Now suppose that S decycles C_n^3 and $|S| = \frac{n-1}{2}$. Note that $n_i \geq 2$, for otherwise the isomorph of K_4 induced by B_i is not decycled. Hence we have

$$2n \leq \sum_{i=0}^{n-1} n_i = 4|S| = 2(n-1)$$

which is a contradiction.

Case 2: $n \equiv 0 \pmod 2$

Let $S = \{0, 1, 2, 4, 6, \ldots, n-2$. Clearly S decycles C_n^2 and has the appropriate cardinality. Now suppose S decycles C_n^3 and $|S| = \frac{n}{2}$. As before, $n_i \geq 2$ for $0 \leq i \leq n-1$, and so

$$2n \leq \sum_{i=0}^{n-1} n_i = 4|S| = 2n$$

Thus $n_i = 2$ for $0 \leq i \leq n-1$. Hence both $(B_i - \{i\}) \cap (V(C_n^3) - S)$ and $(B_{i-3} - \{i\}) \cap (V(C_n^3) - S)$ are nonempty. This means that every vertex in the subgraph induced by $V(C_n^3)$ has degree at least 2, and thus this subgraph contains a cycle, which is a contradiction.

Case 3: $n \equiv 3 \pmod 4$

Let $S = \{0, 1, 2, 4, 5, 8, 9, \ldots, n-4, n-3\}$. This has the requisite cardinality and decycles the graph. Now suppose S decycles C_n^3. If $|\{i, i+1, i+2\} \cap S| \leq 2$ for $0 \leq i \leq n-1$ then consider $j \in V(C_n^3) - S$. Note that j is adjacent to a vertex in each of $\{j+1, j+2, j+3\} - S$ and $\{j-3, j-2, j-1\} - S$. This, as before, means that every vertex in the subgraph induced by $V(C_n^3 - S)$ has degree at least 2, and thus that S does not decycle C_n^3. Thus S contains some three consecutive vertices. Each of the remaining $\frac{n-3}{4}$ vertex disjoint isomorphs of K_4 in C_n^3 must contain at least two vertices, so

$$|S| \geq 3 + 2\left(\frac{n-3}{4}\right) = \frac{n+3}{2}$$

□

Note that it is fairly easy to show using the methods of the final case in the proof of the previous theorem that

$$|S| \geq k + \frac{n-k}{m+1}(m-1)$$

where $n \equiv k \pmod m + 1$ and $0 \leq k \leq n$. This does not, however, give the exact lower bound in every case, although it is certainly better than the lower bound given by Theorem 1. It also seems to be a more difficult problem to find decycling sets of the appropriate cardinalities. However, these methods might work, with appropriate modification, to calculate $\nabla(C_n^m)$ for $m > 3$.

5. References

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